Examples of differentiable mappings into non-locally convex spaces

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Abstract. Examples of differentiable mappings into real or complex topological vector spaces with specific properties are given, which illustrate the differences between differential calculus in the locally convex and the non-locally convex case.[1]

Introduction. Beyond the familiar theories of differentiation in real and complex locally convex spaces ([6], [7]), a comprehensive theory of $C^r_{\mathbb{K}}$-maps between open subsets of topological vector spaces over arbitrary non-discrete topological fields $\mathbb{K}$ has recently been developed [1]. Surprisingly large parts of classical differential calculus remain intact for these maps. For example, being $C^r_{\mathbb{K}}$ is a local property, the Chain Rule holds, and $C^r_{\mathbb{K}}$-maps admit finite order Taylor expansions [1]. Furthermore, when $\mathbb{K}$ is a complete valued field, implicit function theorems for $C^r_{\mathbb{K}}$-maps from topological $\mathbb{K}$-vector spaces to Banach spaces are available [3]. All basic constructions of infinite-dimensional Lie theory (linear Lie groups, mapping groups, diffeomorphism groups) work just as well over general topological fields, valued fields, or at least local fields [4].

In the real locally convex case, the $C^r_{\mathbb{R}}$-maps are precisely the C^r-maps in the sense of Michal and Bastiani (also known as Keller's C^r_c-maps [6]). The Fundamental Theorem of Calculus holds for such maps; in particular, mappings whose differentials vanish at each point have to be locally constant. A mapping into a complex locally convex space is of class $C^\infty_{\mathbb{C}}$ if and only if it is complex analytic in the usual sense (as in [2]). Thus, every $C^\infty_{\mathbb{C}}$-map into a locally convex space is given locally by its Taylor series, and the Identity Theorem holds for such maps. Furthermore, it is known that every $C^1_{\mathbb{C}}$-map into a complete complex locally convex space is automatically of class $C^\infty_{\mathbb{C}}$ (see [1] for all of this).

The purpose of this note is to describe examples showing that these facts become false for mappings with non-locally convex ranges. Thus, for suitable non-locally convex topological vector spaces E, we encounter a smooth injection $\mathbb{R} \to E$ whose derivative vanishes identically; we present a $C^\infty_{\mathbb{C}}$-map $\mathbb{C} \to E$ which is not given locally by its Taylor series around any point; we present a $C^1_{\mathbb{C}}$-map $f\colon \mathbb{C} \to E$ to a metrizable, complete, non-locally convex topological vector space which is not $C^2_{\mathbb{C}}$; and we present a non-zero, compactly supported $C^\infty_{\mathbb{C}}$-map $\mathbb{C} \to E$, the existence of which demonstrates that the Identity Theorem fails for suitable $C^\infty_{\mathbb{C}}$-maps into non-locally convex spaces.

Mappings between open subsets of the field $\mathbb{Q}_p$ of p-adic numbers with similar pathological properties are known in non-archimedian analysis (see [9]). The author also drew inspiration from [8, Ex. II.2.7], where it is shown that the map $[0,1] \to L^{1/2}[0,1]$, $t \mapsto \mathbf{1}_{[0,t]}$ is differentiable at each point (in the ordinary sense), with vanishing derivative.

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Differential calculus. We shall not give the general definition of $C^r_{\mathbb{K}}$-maps between open subsets of topological $\mathbb{K}$-vector spaces here. Rather, we shall work with a simpler definition for the special case of curves (cf. also [9]), which is equivalent to the general definition (cf. [1, Prop. 6.9]). All topological vector spaces are assumed Hausdorff.

Definition. Let $\mathbb{K}$ be a non-discrete topological field, and $f\colon U \to E$ be a mapping from an open subset U of $\mathbb{K}$ to a topological $\mathbb{K}$-vector space E. The map f is said to be *of class* $C^0_{\mathbb{K}}$ if it is continuous; in this case, define $f^{<0>} := f$. We call f *of class* $C^1_{\mathbb{K}}$ if it is continuous and if there exists a continuous map $f^{<1>}\colon U \times U \to E$ such that

$$f^{<1>}(x_1, x_2) = \frac{1}{x_1 - x_2}(f(x_1) - f(x_2))$$

for all $x_1, x_2 \in U$ such that $x_1 \neq x_2$. Recursively, having defined mappings of class $C^j_{\mathbb{K}}$ and associated maps $f^{<j>}\colon U^{j+1} \to E$ for $j = 0, \ldots, k-1$ for some $k \in \mathbb{N}$, we call f *of class* $C^k_{\mathbb{K}}$ if it is of class $C^{k-1}_{\mathbb{K}}$ and if there exists a continuous map $f^{<k>}\colon U^{k+1} \to E$ such that

$$f^{<k>}(x_1, x_2, \ldots, x_{k+1}) = \frac{1}{x_1 - x_2}\left(f^{<k-1>}(x_1, x_3, \ldots, x_{k+1}) - f^{<k-1>}(x_2, x_3, \ldots, x_{k+1})\right)$$

for all $x_1, \ldots, x_{k+1} \in U^{k+1}$ such that $x_1 \neq x_2$. The map f is *of class* $C^\infty_{\mathbb{K}}$ (or *smooth*) if it is of class $C^k_{\mathbb{K}}$ for all $k \in \mathbb{N}_0$.

Here $f^{<k>}$ is uniquely determined, and $f^{<k>}$ is symmetric in its $k+1$ variables. Furthermore, $k!\, f^{<k>}(x, \ldots, x) = \frac{d^k f}{dx^k}(x) =: f^{(k)}(x)$, for all $x \in U$ (cf. [1]).

Example 1. Let $d\nu(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2}\, dx$ be the Gauss measure on $\mathbb{R}$, and $\mu := \nu \otimes \nu$ be the Gauss measure on $\mathbb{C} = \mathbb{R}^2$. Let $E := L^0(\mathbb{C}, \mu)$ be the complex topological vector space of equivalence classes of measurable complex-valued functions on $\mathbb{C}$ (modulo functions vanishing almost everywhere), equipped with the topology of convergence in measure. Thus, a basis of open zero-neighbourhoods of E is given by the sets W_k for $k \in \mathbb{N}$, where W_k consists of those equivalence classes of measurable maps $\gamma\colon \mathbb{C} \to \mathbb{C}$ such that $\mu\left(\{z \in \mathbb{C}\colon |\gamma(z)| \geq \frac{1}{k}\}\right) < \frac{1}{k}$. It is well-known that E is a metrizable, complete, non-locally convex topological vector space, which does not admit any non-zero continuous linear functionals (cf. [5]). Given $\gamma \in L^0(\mathbb{C}, \mu)$ and a closed subset $A \subseteq \mathbb{C}$, we say that γ is *supported in* A if γ vanishes μ-almost everywhere on the complement of A. The *support of* γ is the smallest closed set in which γ is supported. In the following, $\mathbf{1}_A\colon \mathbb{C} \to \{0, 1\}$ denotes the characteristic function of a measurable subset $A \subseteq \mathbb{C}$.

Consider the mapping

$$f\colon \mathbb{C} \to E, \quad f(z) := \mathbf{1}_{A(z)},$$

where $A(z) := \{w \in \mathbb{C}\colon\ \mathrm{Re}(w) \leq \mathrm{Re}(z) \text{ and } \mathrm{Im}(w) \leq \mathrm{Im}(z)\,\}$. Then f has the following properties:

Proposition 1 *$f : \mathbb{C} \to E$ is of class $C^\infty_{\mathbb{C}}$, injective, and $f^{(j)}(z) = 0$ for all $j \in \mathbb{N}$ and $z \in \mathbb{C}$. In particular, f is not given locally by its Taylor series around any point.*

Proof. Apparently f is injective. If we can show that f is $C^\infty_{\mathbb{C}}$, with $f^{(j)}=0$ for all $j\in\mathbb{N}_0$, then, given any $z_0\in\mathbb{C}$, the Taylor series of f at z_0 will only consist of the 0th order term, and hence describes the function which is constantly $f(z_0)$. It therefore does not coincide with the injective function f on any neighbourhood of z_0.

Thus, to complete the proof of the proposition, it suffices to establish the following assertions, by induction on $k\in\mathbb{N}_0$:

(a) f is of class $C^k_{\mathbb{C}}$;

(b) For all $j\in\mathbb{N}$ such that $j\leq k$, we have $f^{(j)}=0$, and, for all $z_1,\ldots,z_{j+1}\in\mathbb{C}$, the element $f^{<j>}(z_1,\ldots,z_{j+1})\in E=L^0(\mathbb{C},\mu)$ is supported in

$$([x_*,x^*]+i\mathbb{R})\,\cup\,(\mathbb{R}+i[y_*,y^*])\,,\quad\text{where}$$

$$\begin{array}{rclrcl} x_* &:=& \min\{\mathrm{Re}(z_1),\ldots,\mathrm{Re}(z_{j+1})\}, & x^* &:=& \max\{\mathrm{Re}(z_1),\ldots,\mathrm{Re}(z_{j+1})\},\\ y_* &:=& \min\{\mathrm{Im}(z_1),\ldots,\mathrm{Im}(z_{j+1})\}, & y^* &:=& \max\{\mathrm{Im}(z_1),\ldots,\mathrm{Im}(z_{j+1})\}\,. \end{array} \tag{1}$$

The case $k=0$. Given $z_1\in\mathbb{C}$, let us show that f is continuous at z_1. To this end, let $z_2\in\mathbb{C}$; define x_*,x^*,y_*,y^* as in (1) (taking $j:=1$). Then the symmetric difference $A(z_1)\oplus A(z_2):=(A(z_1)\setminus A(z_2))\cup(A(z_2)\setminus A(z_1))$ of the sets $A(z_1)$ and $A(z_2)$ is a subset of $([x_*,x^*]+i\mathbb{R})\cup(\mathbb{R}+i[y_*,y^*])$, whence

$$\mu(A(z_1)\oplus A(z_2))\leq|x^*-x_*|+|y^*-y_*|\leq 2\,|z_2-z_1|$$

(using Fubini's Theorem). Note that $f(z_2)-f(z_1)$ is supported in $A(z_1)\oplus A(z_2)$. As the measure of this set tends to 0 as $z_2\to z_1$, we see that $f(z_2)\to f(z_1)$ in E. Thus f is continuous.

Induction step. Let $k\in\mathbb{N}$, and suppose that (a) and (b) hold when k is replaced with $k-1$. Then f is of class $C^{k-1}_{\mathbb{C}}$. In order that f be $C^k_{\mathbb{C}}$, with $f^{(k)}=0$, in view of La. 10.5, La. 10.7 and Prop. 6.2 in [1], we only need to show that

$$g_n:=\frac{1}{z_{n,1}-z_{n,2}}\big(f^{<k-1>}(z_{n,1},z_{n,3},\ldots,z_{n,k+1})-f^{<k-1>}(z_{n,2},z_{n,3},\ldots,z_{n,k+1})\big)\to 0$$

in E as $n\to\infty$, for every sequence $(z_n)_{n\in\mathbb{N}}$ of elements $z_n=(z_{n,1},\ldots,z_{n,k+1})\in\mathbb{C}^{k+1}$ which converges to a diagonal element $(z,z,\ldots,z)\in\mathbb{C}^{k+1}$ for some $z\in\mathbb{C}$, where $z_{n,a}\neq z_{n,b}$ whenever $a\neq b$. Given $n\in\mathbb{N}$, define $x_{n,*}$, x_n^*, $y_{n,*}$ and y_n^* along the lines of (1), using the elements $z_{n,1},\ldots,z_{n,k+1}$ (thus $j=k$). Note that, as a consequence of (b) for k replaced with $k-1$ (valid by the induction hypothesis), each of the elements $f^{<k-1>}(z_{n,2},z_{n,3},\ldots,z_{n,k+1})$, $f^{<k-1>}(z_{n,1},z_{n,3},\ldots,z_{n,k+1})\in E$ is supported in

$$B_n:=([x_{n,*},x_n^*]+i\mathbb{R})\,\cup\,(\mathbb{R}+i[y_{n,*},y_n^*])\,.$$

Hence also g_n is supported in B_n. Since

$$\mu(B_n) \leq 4 \max\{|z_{n,1} - z|, \ldots, |z_{n,k+1} - z|\} \to 0 \quad \text{as} \;\; n \to \infty,$$

we deduce that $\lim_{n\to\infty} g_n = 0$ in E, as required. Thus f is $C^k_{\mathbb{C}}$, and $k!\, f^{<k>}(z, \ldots, z) = f^{(k)}(z) = 0$ for all $z \in \mathbb{C}$.

It only remains to prove the assertion concerning the supports. To this end, let $z_1, \ldots, z_{k+1} \in \mathbb{C}$. If all of $z_1, \ldots, z_{k+1}$ coincide, then $f^{<k>}(z_1, \ldots, z_{k+1}) = 0$ by the preceding, and this is an element with empty support, which therefore is contained in the desired set. Now suppose that $z_a \neq z_b$ for some a, b. By symmetry of $f^{<k>}$ in its $k+1$ variables, we may assume that $z_1 \neq z_2$. Then

$$f^{<k>}(z_1, \ldots, z_{k+1}) = \frac{1}{z_1 - z_2} \left(f^{<k-1>}(z_1, z_3, \ldots, z_{k+1}) - f^{<k-1>}(z_2, z_3, \ldots, z_{k+1}) \right)$$

and, as in the preceding part of the proof, we deduce from the induction hypothesis that this element is supported in the desired set. $\square$

Corollary 1 *Consider E as a real topological vector space. Then $g := f|_{\mathbb{R}} \colon \mathbb{R} \to E$ is an injective $C^\infty_{\mathbb{R}}$-curve whose derivative g' vanishes identically.* $\square$

Example 2. We retain μ and E as in Example 1, but consider now the mapping

$$f \colon \mathbb{C} \to E, \quad f(z) := \mathbf{1}_{A(z)},$$

where $A(z) := \{w \in \mathbb{C} \colon |z| \leq |w| \leq 1\}$. Then f has the following properties:

Proposition 2 *$f \colon \mathbb{C} \to E$ is of class $C^\infty_{\mathbb{C}}$, non-zero, and f has compact support.*

Proof. Clearly $f(z) = 0$ for all $z \in \mathbb{C}$ such that $|z| \geq 1$, entailing that f is compactly supported. Furthermore, $f \neq 0$. Given real numbers $0 \leq r \leq R$, let

$$K(r, R) := \{w \in \mathbb{C} \colon r \leq |w| \leq R\}$$

be the closed annulus with inner radius r and outer radius R in $\mathbb{C}$. Then

$$\mu(K(r, R)) \leq R - r\,. \tag{2}$$

Indeed, we have

$$\begin{aligned} \mu(K(r,R)) &= \int_r^R \int_0^{2\pi} \tfrac{s}{\pi}\, e^{-s^2}\, d\phi\, ds = e^{-r^2} - e^{-R^2} \\ &= (r - R) \cdot (-2\xi\, e^{-\xi^2}) = (R - r)\, 2\xi\, e^{-\xi^2} \leq R - r \end{aligned}$$

for some $\xi \in [r, R]$, using the Mean Value Theorem to pass to the second line, and using that $2te^{-t^2} \leq \sqrt{\frac{2}{e}} < 1$ for all $t \in [0, \infty[$, by an elementary calculation.

The assertion of the proposition will follow if we can prove the following claims, by induction on $k \in \mathbb{N}_0$:

(a) f is of class $C^k_{\mathbb{C}}$;

(b) For all $j \in \mathbb{N}$ such that $j \leq k$, the map $f^{(j)}$ vanishes, and $f^{<j>}(z_1, \ldots, z_{j+1}) \in E$ is supported in the annulus $K(r_*, r^*)$, where

$$r_* := \min\{|z_1|, \ldots, |z_{j+1}|\}, \qquad r^* := \max\{|z_1|, \ldots, |z_{j+1}|\},$$

for all $z_1, \ldots, z_{j+1} \in \mathbb{C}$.

In view of (2), an apparent adaptation of the proof of Prop. 1 establishes these claims. □

Remark. Proposition 2 entails that the identity theorem for analytic mappings becomes invalid when analytic maps are replaced with $C^\infty_{\mathbb{C}}$-maps into non-locally convex spaces.

Example 3. We retain ν and μ as in Example 1 but consider the complex topological vector space $E := L^p(\mathbb{C}, \mu)$ of equivalence classes of complex-valued L^p-functions on $\mathbb{C}$ now, where $p \in \,]\frac{1}{2}, 1[$. Consider

$$f\colon \mathbb{C} \to E, \quad f(z) := \mathbf{1}_{A(z)},$$

where $A(z) := \{w \in \mathbb{C}\colon \mathrm{Re}(w) \leq \mathrm{Re}(z)\}$. The map f has the following properties:

Proposition 3 *$f\colon \mathbb{C} \to E$ is of class $C^1_{\mathbb{C}}$ and f' vanishes, but f is not of class $C^2_{\mathbb{C}}$.*

Proof. Given real numbers $a \leq b$, define $S(a,b) := \{w \in \mathbb{C}\colon a < \mathrm{Re}(w) \leq b\}$. Then

$$\mu(S(a,b)) = \nu(]a,b]) \leq b - a\,. \tag{3}$$

Given $z_1, z_2 \in \mathbb{C}$, where $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ without loss of generality, we have

$$f(z_2) - f(z_1) = \mathbf{1}_{S(\mathrm{Re}(z_1), \mathrm{Re}(z_2))}\,,$$

where $\mu(S(\mathrm{Re}(z_1), \mathrm{Re}(z_2))) \leq \mathrm{Re}(z_2) - \mathrm{Re}(z_1) \leq |z_2 - z_1|$. We easily deduce that f is continuous. Assuming that $z_1 \neq z_2$ here, we have

$$\int_{\mathbb{C}} \left| \frac{f(z_2)(w) - f(z_1)(w)}{z_2 - z_1} \right|^p d\mu(w) = |z_2 - z_1|^{-p} \cdot \mu(S(\mathrm{Re}(z_1), \mathrm{Re}(z_2))) \leq |z_2 - z_1|^{1-p} \to 0$$

as $|z_2 - z_1| \to 0$, showing that $\frac{1}{z_2 - z_1}(f(z_2) - f(z_1)) \to 0$ in E whenever $|z_2 - z_1| \to 0$. Thus $f^{<1>}(z_1, z_2) := 0$ if $z_1 = z_2$, $f^{<1>}(z_1, z_2) := \frac{1}{z_2 - z_1}(f(z_2) - f(z_1))$ if $z_1 \neq z_2$ defines a continuous function $f^{<1>}\colon \mathbb{C} \to E$, showing that f is $C^1_{\mathbb{C}}$ with $f'(z) = f^{<1>}(z,z) = 0$ for all $z \in \mathbb{C}$.

Let $c := \frac{1}{e\sqrt{\pi}}$; then $\frac{1}{\sqrt{\pi}} e^{-t^2} \geq c$ for all $t \in [0,1]$. For $t \in [0, \frac{1}{2}]$, we have

$$\tfrac{1}{t}\left(f^{<1>}(t, 2t) - f^{<1>}(0, 2t)\right) = \tfrac{1}{t^2}\left(\mathbf{1}_{S(t,2t)} - \tfrac{1}{2}\mathbf{1}_{S(0,2t)}\right), \quad \text{whence}$$

$$
\begin{aligned}
&\int_{\mathbb{C}} \left|\tfrac{1}{t}\left(f^{<1>}(t,2t) - f^{<1>}(0,2t)\right)(w)\right|^p \, d\mu(w) \\
&\quad = \left(\tfrac{1}{2t^2}\right)^p \cdot \mu(S(0,2t)) = \left(\tfrac{1}{2t^2}\right)^p \cdot \nu(]0,2t]) \geq \left(\tfrac{1}{2t^2}\right)^p 2t\, c = 2^{1-p} t^{1-2p}\, c \to \infty
\end{aligned}
$$

as $t \to 0$. The map $E \to [0,\infty[$, $\gamma \mapsto \int_{\mathbb{C}} |\gamma(w)|^p \, d\mu(w)$ being continuous, we deduce that $\lim_{t\to 0} \frac{1}{t}\left(f^{<1>}(t,2t) - f^{<1>}(0,2t)\right)$ cannot exist in E. Thus f is not $C^2_{\mathbb{C}}$. $\square$

Corollary 2 *Consider E as a real topological vector space. Then $g := f|_{\mathbb{R}} : \mathbb{R} \to E$ is a $C^1_{\mathbb{R}}$-curve whose derivative g' vanishes identically. However, g is not $C^2_{\mathbb{R}}$.* $\square$

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